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Endogenous Growth: A Sequential Stochastic Search Model for New Technology

Mohammad Hadi Zahedi Vafa*

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Endogenous growth model developed here emphasizes dynamics, with explicit modeling of knowledge accumulation. Considering the uncertainty inherent in any search process, the model presents a dynamic stochastic system in which new technology and capital accumulation are bounded complements—they complement each other to a point, but beyond this the impact of each factor is constrained by the level of the other. As a result, both technological progress and capital accumulation are necessary for sustained growth, but neither on its own is sufficient. Technological advancement stimulates capital accumulation by raising the marginal product of capital. Rapid capital accumulation stimulates R&D investments by raising the expected profitability of innovation. This paper discusses different possible regimes that an economy may find itself in as a result of the interactions between capital accumulation and technological innovations and has important implications for growth-promoting policies, knowledge spillover, and international flow of capital.

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JEL Classification: O41, O31, D9, D83.

1. Introduction

Traditional growth theories often focus exclusively on the accumulation of physical capital as the main determinant of economic growth. Sometimes, labor is also used to explain the economic growth experienced by an economy although the population, and *a fortiori* the labor force, is always assumed to grow exponentially at an exogenous

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*Ph.D in Economics, Assistant Professor, Faculty of Islamic Studies and Economics, Imam Sadiq (عليه السلام) University., Email: zahedi@isu.ac.ir

rate. Modern growth theories-known under the rubric of endogenous growth-on the other hand, relegate the accumulation of physical capital to a less prominent role and consider the accumulation of knowledge or the accumulation of human capital as the fundamental driving force behind economic growth.

Most previous endogenous growth models carry the implication of a scale effect. Since the scale effect is not supported by observed data, these models lose their theoretical underpinnings in the growth literature. Recent growth models have attempted to eliminate the scale effect in R&D. These new models are built in a way that R&D becomes progressively more difficult over time. The challenge faced by economists is to devise a theory of growth in which technical advance, as in endogenous growth theory, and capital formation, and as in neo-classical growth theory, together drive growth.

After Schumpeter it is well accepted that technological progress is driven by firm's self interested attempts to profit from their ingenuity. The innovations do not come from thin air; they are result of costly *search* of the firms, often requiring extensive expenses. The history of search theory goes back to a seminal paper of Stigler (1961) that addressed the question of how large a fixed sample ought to be collected before making the choice of which alternative to accept. Later, Stigler (1970) and McCall (1970) used search theory in labor economics. Other researchers, such as Benhabib and Clive (1983), Morgan and Manning (1985) considered the choice of the set of opportunities to search over at any given time.

In a parallel attempt, Nelson (1961) used the idea of modeling R&D as a search process for a better technology. Evenson and Kislev (1975) developed his idea into a sequential search process. They built a search model in which applied research is a search in a given distribution, and basic research is a shift in the distribution searched. Tesler (1982) used the theory of optimal sequential search to study an industry where firms make homogenous products and incur the expense of research to obtain a new technology that may enable them to lower their costs. Muth (1986) completed the idea of random search for lower cost methods from a fixed population of technological possibilities.

Jovanovic and MacDonald (1994) studied the evolution of a competitive industry with a homogenous product in which a fixed number of firms reduce costs by taking draws from a distribution of untried technologies. As the firm's technologies gradually improve, industry output expands. Bental and Peled (1996) aggregate their model and make the number of draws taken by a firm endogenous. They formulated an endogenous growth model in which technological improvements are generated by a costly and uncertain search process, with search efforts financed by capital whose alternative use is in the production of consumption goods. According to this model, technological progress requires ever-growing investments in R&D, and these growing investments can only be maintained when wealth is growing. The growth of wealth, in turn, is generated by successful R&D efforts through their effects on production technologies. The resulting growth path is characterized by invention cycles reflecting varying incentives to engage in R&D as technology improves.

In this paper, I follow Bental and Peled, *op. cit*., and model R&D as a sequential search for better technologies among a population of technologies characterized by a Pareto distribution. The search is interpreted as a sampling process in which a fixed amount of capital must be expended each time to sample (test) a new technology. In any period, all the firms are assumed to have free access to the best technology employed by any firm in the previous period; that is, it takes one period for a new technology to diffuse completely. Thus a profit-maximizing firm begins each period with a fallback technology, which was the best technology in the preceding period. If the firm chooses not to engage in a sequential search, the fallback technology is the one it will use in the production of the consumption good. On the other hand, if the firm chooses to engage in a sequential search, then every time it pays to sample a new technology, it can either continue or stop the search.

In the model to be developed here, we drop the assumption of sampling with replacement that was used by Bental and Peled, *op. cit*,. Although it might be reasonable, in the context of a job search model, to assume that one can not return to a possibility that has been rejected, it is difficult to justify the assumption that continuing the search means losing temporarily the latest found technology. Because R&D activities involve an accumulation of knowledge and because it is difficult to imagine without good reason that knowledge once discovered can be lost, the assumption that the sequential search is "forgetful" is difficult to maintain. Once the assumption of sampling with replacement is dropped from the model, the dynamic programming argument used to analyze the sequential search must be reworked. We have managed to find a closed-form expression for the equation of the boundary that separates the "continue the search" zone and the "stop the search" zone, and thus obtain an explicit solution to the sequential search problem. The explicit form for the equation of the boundary between "continue" and "stop" the sequential search allows us to conduct a detailed analysis of the transition dynamics, which is missing in Bental and Peled's paper. We also offer some analytical results concerning the asymptotic behavior of the system under sustained growth.

The paper is organized as follows. In Section 2, the one-sector model is presented. The production decisions in each period, after the sequential has been terminated, are analyzed in Section 3. In Section 4, the solution of the sequential search problem is presented. To keep the economic logic more transparent, the technical arguments that are carried out to solve the sequential problem are relegated to Appendix 1 at the end of the paper. The expected lifetime utility maximization of a young individual is presented in Section 5. In Section 6, we classify the various regimes in which the economy might find itself and analyze the transition dynamics that the economy might undergo. Depending on the value of the parameter, the economy might experience sustained growth or not, and this question is analyzed in great detail in this section. In Section 7, we analyze the asymptotic behavior of the system when it experiences sustained growth. Some concluding remarks are given in Section 8.

2. The Model

Time is discrete and denoted by t, $t = 0,1,\ldots$ Three classes of economic agents exist in each period: a young generation, an old generation, and a number of firms that produces a single consumption good.

There is no population growth in the economy. More specifically, in each period the young generation and the old generation are both assumed to have the cardinality of the continuum of measure one. An individual lives two periods and works when he is young. A young individual owns nothing except for one unit of labor that he supplies inelastically on the labor market. He divides his wage between current consumption and saving for his old age. Capital is the only real asset in the economy, and capital investments represent the only possible form of saving. The aggregate saving of the young generation in period t-1 thus constitutes the aggregate capital stock at the beginning of period t.

Firms produce the consumption good from labor and capital. The set of all potential firms is denoted by I, assumed to be a finite set. A firm $i\in I$ begins any period t with a certain level of technological competence—denoted by a_{it}—for producing the consumption good. We shall assume that all the firms begin period t with the same technological level, say $a_{it}=a_t, i\in I$. To operate in period t, the firm must attract savings from the generation of period t-1 by issuing equities on the profits that it generates in period t, and the profits will be distributed to share holders as returns on their savings. This process is assumed to take place at the beginning of each period during a stage called the capital allocation stage. Now because all the firms are assumed to have the same technological level at the beginning of each period, with the same expected return, we have a symmetric equilibrium in which they will all be able to attract the same amount of capital in the capital allocation stage of each period from savers who will be fully and equally diversified between all firms. Thus for each period t, the amount of capital raised by firm i is k_{it} = $K_t/|I|, i\in I$. Here K_t is the aggregate capital stock of the economy at the beginning of period t and $|I|$ represents the number of firms.

Having raised k_{it} units of capital, firm i, $i\in I$, can either combine this amount of capital with hired labor to produce the consumption good, using the technology with which it begins the period, or engage in a sequential search for a new and better vintage of technology. A sequential search involves taking random draws—also called technology draws—from a population of untried technologies. The population of all possible technologies—found or yet to be found—is assumed to consist of all the Cobb-Douglas production functions of the following forms:

$$(1) \qquad y(a,k,l)=ak^{\alpha}l^{1-\alpha}, \qquad (a\geq 1)$$

In (1), $y(a,k,l)$ is the output of the consumption good that is produced from k units of capital and l units of labor; a is the technological level—also called the productivity—of the technology in question; and α, $0<\alpha<1$, is a parameter that characterizes the entire population in which the sequential search is conducted, while the technological level a identifies a specific technology within the population. The following distribution is imposed upon the population of possible technologies.

ASSUMPTION 1: *The probability of drawing a technology with productivity less than or equal to a is given by the following Pareto distribution:*

$$F(a \mid \lambda) = 1 - a^{-\lambda}, \qquad (1 \le a < \infty),$$

where λ is a positive parameter. Observe that for any given $a \ge 1$, $F(a \mid \lambda)$ is increasing in λ; that is, the cumulative probability of finding a technology with productivity at least as high as a declines as λ increases. In other words, the Pareto distribution is stochastically decreasing in its parameter: a higher value for the parameter implies a less favorable population of potential technologies in which the sequential search is conducted.

In any period, a firm finances its sequential search from part of the capital it raised during the capital allocation stage. There is no fixed cost in the sequential search and the firm can take any number of technology draws at the cost of ε units of capital per draw. A technology draw completely reveals the productivity of the technology found. The maximum numbers of technology draws that firm i can take in period t is $\max\{0, [[k_{it} / \varepsilon]]\}$, where $[[x]]$ is defined to be the greatest integer less than or equal to x. The sequential nature of the search can be described as follows. Suppose that firm i starts the sequential search and finds a technology with productivity a^1 after the first draw. At this point in time, it has $k_{it} - \varepsilon$ units of capital left and the best technology under its command has a productivity equal to $\max\{a_{it}, a^1\}$. The firm might decide to stop or continue the search. If it continues the search, let a^2 be the productivity of the technology found in the second draw. After the second draw, the firm has $k_{it} - 2\varepsilon$ units of capital left and the best technology at its disposal has a productivity equal to $\max\{a_{it}, a^1, a^2\}$. At any instant during the sequential search, the decision whether to stop or continue the search depends on how much capital remains and the best technology at its disposal.

The productivity of the technologies drawn in a sequential search are assumed to be independently and identically distributed. This assumption can be justified by noting that under the distribution F each productivity level $a \in [1, \infty)$ has probability zero. Furthermore, to rule out the possibility of strategic behavior by the firms during the search stage of any period, the technologies drawn by a firm in its sequential search are assumed to be observed only by the firm itself. Finally, to ensure that a firm will develop and use a technology it has found instead of selling it to another firm, and that it will engage in a sequential search—whenever profitable—the capital owned by a firm is assumed to become firm-specific.

Let $\hat{a}_{it}$ and $\hat{k}_{it}$ denote, respectively, the productivity of the best technology under the command of firm i and its remaining capital when this firm enters the production stage of period t. If the firm did not engage in a sequential search, then $\hat{a}_{it} = a_{it}$ and $\hat{k}_{it} = k_{it}$. On the other hand, if it engaged in a sequential search and took n technology draws, then $\hat{a}_{it} = \max\{a_{it}, a^1, \ldots, a^n\}$ where $a^1, \ldots, a^n$ are the productivity of the technologies found in the search, and $\hat{k}_{it} = k_{it} - n\varepsilon$. The technology with which firm i enters the production function stage of period t is the given by the following short-run production function:

$$(2) \qquad y(\hat{a}_{it}, \hat{k}_{it}, l) = \hat{a}_{it}\hat{k}_{it}^{\alpha} l^{1-\alpha}.$$

For each $t = 0,1,\ldots,$ the state of the economy at the beginning of period t is represented by the vector (a_t, K_t), where a_t is the common technological level of all the firms and K_t is the aggregate capital stock – all at the beginning of period t. The initial state of the economy, namely (a_0, K_0) is assumed to be known.

A firm might decide to engage in a sequential search to improve its productivity. Although such a firm begins the search stage with a known technological level a_{it} and a known capital stock k_{it}, the random nature of the search means that $\hat{a}_{it}$, the highest technological level, and $\hat{k}_{it}$, the remaining capital, under its command for producing the consumption good are only known when the search ends. Hence the profit generated by the firm is not known at the time it begins the search. However, once $\hat{a}_{it}$ and $\hat{k}_{it}$ are realized, the uncertainty about the production technology of firm *i* is completely resolved and is as given in (2).

Because the old generation in each period are the owners of the firms in that period, and because those individuals will not be alive in the following period, all that they care about is the current profit of the firms, not their future profitability. Furthermore, with full depreciation, the firms do nothing this period to affect its future and are essentially single period entities. Thus the firms will pursue the objective of maximizing profits in each period. Given this objective, and after $\hat{a}_{it}$ and $\hat{k}_{it}$ have been realized, firm *i* will combine $\hat{k}_{it}$ with hired labor to maximize the short-run profit in the production of the consumption good. The demand for labor by all the firms will then determine the realized equilibrium wage rate in period *t*. Once the equilibrium wage rate is known, a young individual in period *t* can decide on his current consumption and his saving. The aggregate saving of the young individuals of period *t* then constitute K_{t+1}, the aggregate capital stock at the beginning of period $t + 1$. We shall assume that it takes one period for the best technology at the end of the sequential search of period *t* to diffuse throughout the economy. Hence the common technological level with which all the firms begin the next period is given by $a_{t+1} = \max_{i \in I} \hat{a}_{it}$. The state of the economy at the beginning of period t+1 is then represented by the vector (K_{t+1}, a_{t+1}), and the process just explained for period *t* repeats itself in period t+1, driving the system to a new state (K_{t+2}, a_{t+2}) in period t+2, and so forth.

The structure of the equilibrium in each period is that of an equilibrium extending over three successive stages—the capital allocation stage, the sequential search strategy, and the production stage. The strategy for analyzing such equilibrium is by backward induction. We thus begin by analyzing the last stage.

The model thus describes the evolution of a market of a homogenous product. The supply side of the model comprises a fixed number of price-taking firms that maximize profits. To operate at time *t*, firms attract savings at period t-1 by issuing equities on time *t* profits. Each firm makes decision on behalf of shareholders, whether to invest in production with a known technology or to invest in R&D

with uncertain payoffs. The firm's profits are distributed to shareholders as returns on their savings. There is no entry or exit. In a symmetric equilibrium, all firms are identical at the beginning of each period. This symmetry is motivated by the absence of any long-lasting impact of any action taken by the firms in any period. Consequently, the distribution of returns offered by all the firms in each period will be the same, and all risk-averse agents will be fully and equally diversified across firms. Provided we recognize that the firm's costs include the necessary risk premium, the desire by savers to spread their risks between many firms means that each individual firm will face a rising cost of capital. In this way the number of firms would be endogenous, all beginning their operations in any period with same amount of capital, pursuing the same search strategy. Since solving for capital market equilibrium and finding the endogenous number of firms are not our primary objectives in this paper, we leave this important extension for future research.

3. The Production Stage

In the production stage of period t, an operating firm i produces the consumption good according to the technology represented by the short-run production function (2). Let w be the wage rate, and assume that w is taken as given by all the operating firms. Firm i solves the following profit maximization:

$$\max_l \left(\hat{a}_{it}\hat{k}_{it}^{\alpha} l^{1-\alpha} - wl\right) \tag{3}$$

The solution of (3) yields the following demand for labor by firm i:

$$l(\hat{a}_{it}, \hat{k}_{it}, w) = \left[\frac{(1-\alpha)\hat{a}_{it}}{w}\right]^{1/\alpha} \hat{k}_{it} \tag{4}$$

Using (4) in (2), we obtain the following expression for the output of firm i:

$$y(\hat{a}_{it}, \hat{k}_{it}, l(\hat{a}_{it}, \hat{k}_{it}, w)) = \hat{a}_{it}^{\;1/\alpha}\hat{k}_{it}\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} \tag{5}$$

The profit (i.e. returns to capital) generated by firm i is:

(6) $$\pi(\hat{a}_{it},\hat{k}_{it},w)=\alpha(\hat{a}_{it}^{1/\alpha})\hat{k}_{it}\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha}$$

Summing (4) over $i\in I$, then equating the result to 1, we obtain the following market-clearing condition for labor:

(7) $$\left(\frac{1-\alpha}{w}\right)^{1/\alpha}\left(\sum_{i\in I}\hat{a}_{it}^{1/\alpha}\hat{k}_{it}\right)=1$$

The realized equilibrium wage rate is then given by:

(8) $$\hat{w}((\hat{a}_{it},\hat{k}_{it})_{i\in I_t})=(1-\alpha)\left(\sum_{i\in I}\hat{a}_{it}^{1/\alpha}\hat{k}_{it}\right)^{\alpha}$$

Substituting the right side of (8) for w in (5), we obtain the following expression for the output of the firm i in terms of the data $(\hat{a}_{i't},\hat{k}_{i't})_{i'\in I}$

(9) $$\hat{y}_i((\hat{a}_{i't},\hat{k}_{i't})_{i\in I})=(\hat{a}_{it}^{1/\alpha})\hat{k}_{it}\left(\sum_{i'\in I}\hat{a}_{i't}^{1/\alpha}\hat{k}_{i't}\right)^{\alpha-1}.$$

Similarly, substituting the right side of (8) for w in (6), we obtain the following expression for the profit of the firm i in terms of the data $(\hat{a}_{i't},\hat{k}_{i't})_{i'\in I}$

(10) $$\hat{\pi}_i((\hat{a}_{i't},\hat{k}_{i't})_{i'\in I})=\alpha(\hat{a}_{it}^{1/\alpha})\hat{k}_{it}\left(\sum_{i'\in I}\hat{a}_{i't}^{1/\alpha}\hat{k}_{i't}\right)^{\alpha-1}$$

Summing (9) over $i\in I$, we obtain the following expression for national income in terms of the data $(\hat{a}_{it},\hat{k}_{it})_{i\in I}$:

(11) $$\hat{Y}((\hat{a}_{it},\hat{k}_{it})_{i\in I_t})=\left(\sum_{i\in I_t}\hat{a}_{it}^{1/\alpha}\hat{k}_{it}\right)^{\alpha}$$

Similarly, summing (11) over $i\in I$, we obtain the following expression for the gross income of the old generation of period t:

$$\sum_{i\in I}\hat{\pi}_i((\hat{a}_{i't},\hat{k}_{i't})_{i'\in I_t}) = \alpha\left(\sum_{i\in I}\hat{a}_{it}^{1/\alpha}\hat{k}_{it}\right)^{\alpha} \tag{12}$$

The realized gross rate of return to the capital investment received by an old individual of period t is thus given by:

$$\hat{r}((\hat{a}_{it},\hat{k}_{it})_{i\in I_t}\mid K_t) = \alpha\left(\sum_{i\in I_t}\hat{a}_{it}^{1/\alpha}\hat{k}_{it}\right)^{\alpha} / K_t \tag{13}$$

4. The Search Stage

Consider a firm, say i, that enters the search stage of period t with $k_{it} > 0$ as the amount of capital it raised from the young generation of period t-1. This firm has to decide whether to engage in a sequential search to find a new and better technology than the one at its disposal when the period begins.

Now the profit of firm *i,* as represented by (6), depends on the technological level and the remaining capital under its command when it enters the production stage as well as the wage rate that prevails in this stage. Furthermore, according to (8), the realized equilibrium wage rate depends on the technological levels and the remaining capital stocks of all the firms—including firm *i*—when the production stage begins. Thus strictly speaking, the search strategy chosen by a firm will have an impact on the realized equilibrium wage rate. A complete modeling of this impact however is quite involved.

A rational expectation equilibrium in the search stage consists of a distribution, say $G(w)$, of the wage rate and a search strategy for each operating firm. Two consistency conditions are imposed by such a rational expectations equilibrium. First, the search strategy pursued by each operating firm must be optimal, given that $G(w)$ represents its expectations about the equilibrium wage rate. Second, when these search strategies are carried out, the resulting equilibrium wage rate, as represented by (8), has $G(w)$ as its distribution.

Given the expectations that firm i holds about the equilibrium wage rate, we can find its optimal sequential search with the help of dynamic programming by carrying out an induction on the maximum numbers of technology draws that the firm can take. The following result is established and a detailed proof is in Appendix 1.

PROPOSITION 1: *Let* $\bar{k}: a \rightarrow \bar{k}(a), a \geq 1,$ *be the curve defined by* $\bar{k}(a) = \varepsilon(1+(\alpha\lambda - 1)a^{\lambda})$
and $\bar{a}: k \rightarrow \bar{a}(k), k \geq 0,$ *be the curve defined as follows*:
$\bar{a}(k) = 1$ *for* $0 \leq k \leq \varepsilon\alpha\lambda,$
= *the inverse of* $\bar{k}(a)$ *for* $k > \varepsilon\alpha\lambda$, *i.e.,* $\bar{a}(k)$ *is the value of a defined implicitly by* $\bar{k}(a) = \varepsilon(1+(\alpha\lambda - 1)a^{\lambda})$.
Next, let (*a, k*) *be the current state of the sequential search of a firm. If* $a < \bar{a}(k)$, *then the firm should continue the search; otherwise, it is optimal for the firm to stop the search. As defined,* $\bar{a}(k)$ *represents the threshold technological level for the firm in terms of its remaining capital k such that if the best technology currently at the disposal of the firm falls short of this threshold level, then it is optimal for the firm to continue the search.* The following figure depicts the relationship between *k* and $\bar{a}(k)$.

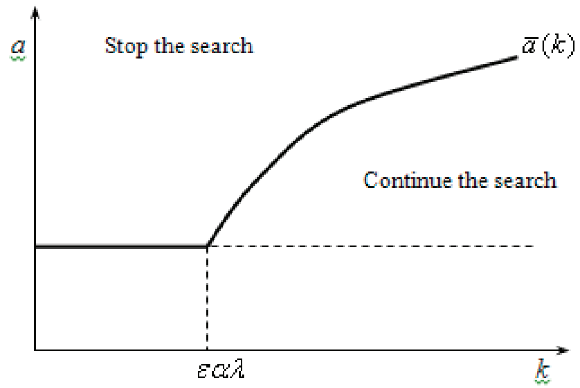

Figure 1. The threshold technological level $\bar{a}(k)$

Given the initial condition (a_{it}, k_{it}), the search strategy described in Proposition 1 completely determines the distribution of $(\hat{a}_{it}, \hat{k}_{it})$, and in particular the distribution of $\hat{\zeta}_{it} = \hat{a}_{it}^{1/\alpha}\hat{k}_{it}$, the random variable that embodies simultaneously the productivity and the remaining capital of firm *i* at the beginning of the production stage in period *t*.

5. Lifetime Utility Maximization

The preferences of an individual are assumed to satisfy the following assumption.

ASSUMPTION 2: The lifetime utility function of a young individual depends only on his current and future consumption, and is assumed to have the following form:

$$u(c^0, c^1) = \log c^0 + \delta \log c^1 ,$$

where $0<\delta<1$ is the discount factor; c^0 is the consumption when he is young; and c^1 is the consumption when he is old.

Let w be the wage rate received by a young individual. If c^0 is his current consumption, then his saving is $w - c^0$. This individual solves the following expected lifetime utility problem:

$$\max_{c^0} \left[\log c^0 + \delta \mathrm{E} \log(\tilde{r}(w - c^0))\right],$$

where $\tilde{r}$ is a random variable representing his anticipated gross rate of return to capital investment and E is the expectation operator with respect to the distribution of $\tilde{r}$. The solution of this expected lifetime utility problem yields the following saving function:

$$w - c^0 = \frac{\delta w}{1+\delta},$$

which asserts that a young individual always saves a constant fraction of his wage. Thus if w_t is the wage rate in period *t,* then the aggregate capital stock at the beginning of the next period is

$$K_{t+1} = \frac{\delta w_t}{1+\delta}.$$

6. Classification of Regimes and the Transition Dynamics

Intuitively, we expect that the economy will experience sustained growth if R&D activities are highly productive. Now R&D activities are profitable if the share of capital in national income is high, and the population from which technologies are sampled are promising. In terms of the parameters of our model, these requirements translate into a high value of α and a low value of λ, or equivalently, a high value of $1/(1-\alpha)$ and a low value of λ.

In this section, we show that if $1/(1-\alpha) < \lambda$, then there is no sustained growth and the economy will converge to a stationary equilibrium in the long run. This result, which is a strengthened version of Proposition 4.4 in Bental and Peled, *op cit.*, is stated as our Proposition 2.

To analyze the dynamic behavior of the economy, we begin with its evolution through time when there is no R&D. To this end, let K_t be the aggregate capital stock at the beginning of period *t.* When there is no R&D, the technological level of the economy is constant, say $a_t = a$ for $t = 0, 1, \ldots$ If we denote the aggregate output in period *t* by Y_t, then because the aggregate labor input is $L = 1$, we must have

(14) $$Y_t = aK_t^{\alpha}$$

and by using (4), we obtain the following expression for the aggregate capital stock of the next period

(15) $$K_{t+1} = \frac{\delta}{1+\delta}(1-\alpha)aK_t^{\alpha}$$

Equation (15) asserts that If the economy chooses never to conduct R&D, then as Solow's model the aggregate capital stock converges geometrically to the stationary level $(\frac{\delta(1-\alpha)a}{1+\delta})^{(1/(1-\alpha))}$, the following figure depicts the stationary equilibrium level of the aggregate capital stock as a function of the technological level *a* of the economy, given that no more R&D activities will be undertaken once *a* has been attained.

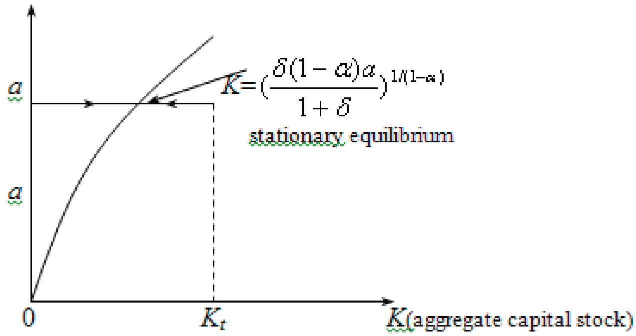

Figure 2. Stationary equilibrium level of the aggregate capital stock

An alternative representation of the economy in its convergence to the stationary equilibrium depicts the equilibrium level of capital per firm as a function of the technological level *a*, given that no more R&D activity will be undertaken by the firms. The curve $\overset{o}{k}: a \to \frac{1}{|I|}\left[\frac{\delta(1-\alpha)a}{1+\delta}\right]^{1/(1-\alpha)}$ represents the stationary equilibrium capital stock per firm as a function of the economy's technological level *a,* given that no more R&D activities are undertaken. The dynamics of the economy is completely determined by the relative position of the curves $\bar{k}(a) = \varepsilon(1+(\alpha\lambda-1)a^{\lambda})$ and $\overset{o}{k}(a) = \frac{1}{|I|}\left[\frac{\delta(1-\alpha)a}{1+\delta}\right]^{1/(1-\alpha)}$. For each value of *a* on the vertical axis, the horizontal distance of the curve $\bar{k}: a \to \varepsilon(1+(\alpha\lambda-1)a^{\lambda})$ relative to that of the curve $\overset{o}{k}: a \to [(\frac{\delta(1-\alpha)a}{1+\delta})^{1/(1-\alpha)}]/|I|$, is given by the following ratio:

$$(16)\quad \frac{\varepsilon(1+(\alpha\lambda-1)\alpha^{\lambda})}{\left(\frac{\delta(1-\alpha)a}{1+\delta}\right)^{1/(1-\alpha)}/|I|} = \frac{\varepsilon|I|}{\left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1/(1-\alpha)}}\left(a^{-1/(1-\alpha)} + (\alpha\lambda-1)a^{\lambda-1/(1-\alpha)}\right)$$

We shall call (16) the horizontal distance of $\bar{k}$ relative to $\overset{o}{k}$. Observe that if $\lambda < 1/(1-\alpha)$, then the right side of (16) declines monotonically from $\varepsilon\alpha\lambda|I|/\left(\frac{\delta(1-\alpha}{1+\delta}\right)^{1/(1-\alpha)}$ to zero as $a \to +\infty$. If

$\lambda = 1/(1-\alpha)$, then it declines monotonically from $\varepsilon\alpha\lambda\,|\,I\,|\,/\left(\dfrac{\delta(1-\alpha}{1+\delta}\right)^{1/(1-\alpha)}$ to $\varepsilon\alpha\lambda\,|\,I\,|\,(\alpha\lambda-1)/\left(\dfrac{\delta(1-\alpha)}{1+\delta}\right)^{1/(1-\alpha)}$ when $a \rightarrow +\infty$. When $\lambda > 1/(1-\alpha)$, the right side of (16) tends to ∞ as $a \rightarrow +\infty$. Furthermore, depending on the value of α and λ, the horizontal distance of $\bar{k}$ relative to $\overset{o}{k}$ might rise monotonically, or it might first decline, reach a minimum, then rises monotonically to $+\infty$ as a rises from 1.

6-1. The case 1/(1-α)<λ

When $1/(1-\alpha) < \lambda$, the horizontal distance of $\bar{k}$ relative to $\overset{o}{k}$ tends to 0 as $a \rightarrow +\infty$, i.e., $\overset{o}{k}$ will be to the left of $\bar{k}$ when a is large. There are four possibilities to consider: (i) $\overset{o}{k}$ completely on the left of $\bar{k}$, (ii) $\overset{o}{k}$ on the left of and tangent to $\bar{k}$, (iii) $\overset{o}{k}$ crossing $\bar{k}$ at two points, and (iv) $\overset{o}{k}$ crossing $\bar{k}$ at exactly one point. By working through the dynamics of model for each of these possibilities we can derive the following proposition.

PROPOSITION 2: *If* $1/(1-\alpha) < \lambda$*, then depending on the configuration of the curve* $\overset{o}{k}$ *and the curve* $\bar{k}$*, the economy might engage in R&D for some periods. However, in the long run all growth stops and the economy converges to a stationary equilibrium.*

6-2. The case λ = 1/(1-α)

Under this case there are three possibilities. By working through these possibilities we can derive the following proposition.

PROPOSITION 3: *Suppose that* $1/(1-\alpha) = \lambda$*. There are three possibilities to consider*

i. If $\frac{\varepsilon|I|(\alpha\lambda-1)}{\left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1/(1-\alpha)}} \geq 1$, *then the economy might engage in R&D at most in one period. In the long run, the system converges to a stationary equilibrium.*

ii. *If* $\frac{\varepsilon|I|(\alpha\lambda-1)}{\left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1/(1-\alpha)}} < 1$, *then the economy will experience sustained growth.*

iii. *If* $\frac{\varepsilon|I|(\alpha\lambda-1)}{\left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1/(1-\alpha)}} < 1 < \frac{\varepsilon|I|\alpha\lambda}{\left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1/(1-\alpha)}}$, *then the system might engage in R&D in short run. However, in the long run growth stops and economy converges to a stationary equilibrium.*

6-3. The case $1/(1-\alpha) > \lambda$

When $1/(1-\alpha) > \lambda$, the curve $\overset{o}{k}(a)$ will be to the right of the curve $\bar{k}(a)$ when a is large. There are two possibilities to consider. First, the possibility that the curve $\overset{o}{k}(a)$ is on the left of the curve $\bar{k}(a)$ at the beginning, when a rises from 1 then cross the latter curve from above. This case occurs when $\varepsilon\alpha\lambda > \left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1/(1-\alpha)} / |I|$. Second possibility occurs when $\varepsilon\alpha\lambda \leq \left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1/(1-\alpha)} / |I|$. Under this possibility, the curve $\overset{o}{k}(a)$ is completely on the right of the curve $\bar{k}(a)$. And there is sustained growth in the long run no matter what the initial condition (a_0, k_0) is. In this case the productivity of R&D is high or the cost of search is low.

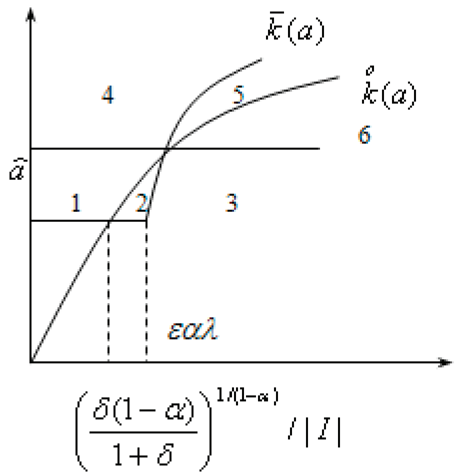

Figure 3.

PROPOSITION 4: *Suppose that* 1/(1-α)>λ. *There are two possibilities to consider*

i. *If* $\varepsilon\alpha\lambda > \left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1/(1-\alpha)} /|I|$, *then the configuration of* $\overset{o}{k}(a)$ and $\bar{k}(a)$ *as depicted in Figure* 3 *the system will experience sustained growth if initial stage is in Regions 4, 5, or 6. On the other hand, if the initial stage of system is in Regions 1 or 2, no R&D will carry out. Under this scenario the system will converge to stationary equilibrium also it has a potential to grow indefinitely. If the state of system in Region 3 the R&D activity will carry out.*

ii. *If* $\varepsilon\alpha\lambda \le \left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1/(1-\alpha)} /|I|$. *Then economy will experience sustained growth.*

7. The Asymptotic Behavior of the System under Sustained Growth ($1/(1-\alpha) > \lambda$)

Let $\hat{a}$ be the technological level such that the economy will enter the zone of sustained growth once its technological level rises above $\hat{a}$. We shall now study the behavior of the economy once it enters this zone.

For each $a \geq 1$, let $(k^n(a))_{n=0}^{\infty}$ be the sequence defined recursively as follows. For $n = 0$, set $k^n(a) = \bar{k}(a) = \varepsilon(1+(\alpha\lambda-1)a^{\lambda})$. For $n \geq 0$, $k^{n+1}(a)$ is the unique value of the k that satisfies the following relation:

$$\left(\frac{\delta(1-\alpha)}{1+\delta}\right)a(|I|k^{n+1})^{\alpha}/|I| = k^n(a), \qquad (n = 0,1,\ldots),$$

As defined, $(a, k^n(a))$, $n = 0,1,\ldots$, is the state of the economy in the next period, given that $(a, k^{n+1}(a))$ is the state in the current period. Now we follow an induction to compute $k^n(a)$. We have

$$k^n(a) = \left[\frac{\varepsilon|I|^{1-\alpha^n}}{\left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1+\alpha+\cdots+\alpha^{n-1}}} \frac{(1+(\alpha\lambda-1)a^{\lambda}}{a^{1+\alpha+\cdots+\alpha^{n-1}}}\right]^{1/\alpha^n}.$$

Using the definition of $k^n(a)$ and $k^{n+1}(a)$, we can write

$$k^{n+1}(a) = \left[\frac{\varepsilon|I|^{1-\alpha^{n+1}}}{\left(\frac{\delta(1-\alpha)}{1+\delta}\right)^{1+\alpha+\cdots+\alpha^{n}}} \frac{(1+(\alpha\lambda-1)a^{\lambda}}{a^{1+\alpha+\cdots+\alpha^{n}}}\right]^{1/\alpha^{n+1}}, \qquad (n = 0,1,\ldots,).$$

Now using the assumption $1/(1-\alpha) = 1+\alpha+\alpha^2+\cdots > \lambda$, we can assert the existence of a positive integer, say $n(\alpha, \lambda)$, such that

$$\text{(17)} \qquad \begin{aligned} 1+\alpha+\alpha^2+\cdots+\alpha^n &< \lambda \quad \textit{if} \quad n \leq n(\alpha,\lambda), \\ 1+\alpha+\alpha^2+\cdots+\alpha^n &> \lambda \quad \textit{if} \quad n > n(\alpha,\lambda). \end{aligned}$$

Using (17), we can then assert that when $n > n(\alpha,\lambda)$, we must have $\lim_{a\to\infty} k^{n+1}(a) = 0$.

If (a_t, k_t), the state of economy at the beginning of period t, $t = 0$, $1,\ldots$, satisfies $k^n(a_t) < k_t \leq k^{n-1}(a_t)$, then R&D activities will not be undertaken in period t, t+1, …, t+n-1; the sequential search will only be resumed in period $t + n$.

Suppose that the economy is on a sustained growth path. For each period t = 0, 1,…, let $(\hat{a}_{it}, \hat{k}_{it})_{i \in I}$ be the state of the economy at the end of the sequential search in period t. The state of the economy at the beginning of period $t+1$ is then given by

$$(a_{t+1}, Y_{t+1}) = (\max_{i \in I} \hat{a}_{it}, (\sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it})^{\alpha}).$$

The state of a firm in period t +1 is thus

$$(a_{t+1}, k_{i,t+1}) = (\max_{i \in I} \hat{a}_{it}, \frac{\delta(1-\alpha)}{(1+\delta)|I|}(\sum_{i \in I} \hat{a}_{it}^{1/\alpha} \hat{k}_{it})^{\alpha}).$$

Because the economy is on a sustained growth path, both a_t and k_{it} tend to infinity as t tends to infinity. Furthermore, because for any $n \geq n(\alpha, \lambda), k^n(a)$ is a strictly decreasing to zero as a tends to infinity, the state of a firm at any point in time on the sustained growth path will remain on the right of the curve $k^n : a \to k^n(a)$, i.e., $k_{it} > k^n(a_t)$ for all t greater than a certain positive integer. Hence the number of periods during which the economy grows without engaging in R&D is bounded above by $n(\alpha, \lambda)$. Now for a given value of k_{it}, an increase in a_t will move the state (a_t, k_{it}) vertically upward, inducing a longer phase of growth without R&D. On the other hand, an increase in k_{it} , with a_t maintained at the same level, induces a shortening of such a phase. A successful search in period t-1 implies a rise in both a_t and k_t and it is not clear whether the search will be resumed sooner or later. We summarize the results just discussed in the following proposition.

PROPOSITION 5: *Suppose that* $1/(1-\alpha) > \lambda$. *Also, suppose that the economy is in the region of sustained growth path; that is, at a certain point in time, say t, the technological level* a_t *exceeds the critical value* $\hat{a}$, *defined at the beginning of Section* 7. *Then along the sustained growth path, the length of each phase during which the economy grows without engaging in R&D is bounded above by the positive integer defined by* (17). *It is not clear whether a more* *successful search will lengthen the no R&D phase that begins in the next period.*

Consider a technological level a and let c be the point (a, $\varepsilon(1+(\alpha\lambda-1)a^{\lambda})$. If c is the state of the economy at the beginning of the current period, then the state of the economy at the beginning of the following period is the point

$$c' = (a, \frac{\delta(1-\alpha)a}{|I|(1+\delta)}(|I|\varepsilon(1+(\alpha\lambda-1)a^{\lambda}))^{\alpha}.$$

If the point b =(a, $\varepsilon(\alpha\lambda-1)a^{\lambda}$) represents the state of the economy at the beginning of the current period, then the state of the economy at the beginning of the following period is represented by the point

$$b' = (a, \frac{\delta(1-\alpha)a}{|I|(1+\delta)}(|I|\varepsilon(\alpha\lambda-1)a^{\lambda})^{\alpha}).$$

The horizontal distance of point c relative to point b' is

$$\frac{\varepsilon(1+(\alpha\lambda-1)a^{\lambda})}{\frac{\delta(1-\alpha)}{|I|^{1-\alpha}(1+\delta)}(\varepsilon(\alpha\lambda-1))a^{\alpha\lambda+1}}$$

In the present section, it is assumed that $1/(1-\alpha)>\lambda$, i.e., $\lambda-\alpha\lambda-1<0$. Hence this ratio tends to 0 as a tends to infinity. The distance cb'—and *a fortiori* the distance bb'—thus tends to infinity as a tends to infinity. The horizontal distance between point b'and point c' is

$$\frac{\delta(1-\alpha)}{|I|^{1-\alpha}(1+\delta)}(\varepsilon(\alpha\lambda-1))^{\alpha}a^{\alpha\lambda+1}((1+1/(\alpha\lambda-1)a^{\lambda})^{\alpha}-1),$$

which can be shown it tends to infinity as a tends to infinity.

Now consider a period t in which R&D activities are carried out and suppose that the state of economy at the beginning of this period is (a_t, k_{it}). If the R&D activities of the firms do not yield any technology better than a_t, then the state of a typical firm, say i, at the beginning of the production stage of period t is represented by the point $(\hat{a}_{it}, \hat{k}_{it}) = (a_t, \hat{k}_{it})$, a point inside the interval (b, c]. The state of the economy at the beginning of period t+1 is (a_{t+1}, k_{t+1}), which belongs to the interval (b', c']. If the R&D activities carried out by the firms in period t+1 are a complete failure, the state of the system at the beginning of the production phase will again lie in the interval (b, c], which in turn implies that the state of economy at the beginning of period t+2 will again belong to the interval (b', c']. This process might persist for a number of periods, and the economy's technological level remains at a_t during this phase, although the capital stock per firm during periods might vary between the horizontal distance of b' and the horizontal distance of c'. During such a phase, the minimum number of technology draws taken by a firm is $Ceiling\left[\frac{d(c,b')}{\varepsilon}\right]$, where $d(c, b')$ is the distance between point c and point b', and $Ceiling[x]$ is the smallest integer greater than or equal to x. Also, the maximum number of technology draws taken by a firm is $Ceiling\left[\frac{d(c,c')}{\varepsilon}\right]$, where $d(c, c')$ is the distance between points c and points c'. We already shown that $d(c, b')\to\infty$ and $d(b', c') \to\infty$ as $a\to\infty$. Hence for an advanced economy (i.e., an economy with a high value of a) more and more resources will be devoted to R&D, actually without improving the production capacity of the economy. Under such a scenario, the probability of finding no better technology by a firm in a period is bounded above by

$$(1 - a^{-\lambda})^{Ceiling\left[\frac{d(c,b')}{\varepsilon}\right]} \tag{18}$$

and bounded below by

$$(1 - a^{-\lambda})^{Ceiling\left[\frac{d(c,c')}{\varepsilon}\right]} \tag{19}$$

It can be shown that (18) is strictly decreasing to zero when $a \to +\infty$. Thus as the economy becomes more advanced, the probability of failing to find a better technology tends to decrease. We summarize the results just discussed in the following proposition.

PROPOSITION 6: *Suppose that* $1/(1-\alpha) > \lambda$. *Also, suppose that the economy is in the region of sustained growth. Then along a sustained growth path, more and more resources are devoted to R&D. Furthermore, the ever increase amount of resources devoted to R&D more then compensates for the diminishing return to R&D, and the end result is that the probability of success in discovering new and better technologies tends to one when the technological level* $a \to +\infty$.

8. Conclusion

In this chapter, we have presented and analyzed exhaustively a one-sector model of endogenous growth in which firms can improve their productivity by engaging in a sequential search for a new technology. In the model, the sequential search, whose outcomes are random, is interpreted as a process of testing new and untried technologies. The testing, also called taking technology draws, require the expenditure of cumulable capital, say laboratory equipment. To operate in any period, a firm must raise capital from the young generation of the previous period. Part of the capital raised by a firm at the beginning of every period is spent in the sequential search; what is left of the capital it raised will be combined with labor according to the best technology at its disposal at the end of the search to produce the consumption good.

The new and untried technologies as well as those already found are assumed to come from a population of Cobb-Douglas technologies using labor and capital as inputs. The technologies in this population all have the same elasticity of output with respect to the capital input and what differentiates one member of the population from another is the difference in their technological levels, which are assumed to follow a Pareto distribution. The Pareto distribution that characterizes the technological levels of the technologies in the population is unbounded from above and thus the probability of finding a technology with a technological level higher than the one currently available is always positive. There is thus a possibility for sustained and unending growth. Another property of the Pareto distribution is that it exhibits diminishing returns: the higher is the current technological level, the harder it is to find a new and better technology using the same amount of cumulable capital as input in the sequential search process. Such a distribution thus allows for the possibility of growth while not contradicting the empirical evidence of diminishing returns to R&D.

As the aggregate capital stock increases and the technological level of the economy remains the same, diminishing returns will set in. There comes a point when the marginal product of capital, given a constant labor force, is so low that it is better to use up part of the capital already accumulated to find a new and better technology. If the sequential search is successful and a new technology with a higher technological level has been found, the search is temporarily suspended. Production is now carried out using the latest technology, and more capital will be accumulated. Again, even with the new and better technology, there comes a point in time when diminishing returns reappear and a new sequential search is called for. The model thus exhibits cycles constituting of a phase during which R&D activities are undertaken to be followed by periods during which production model is carried out using the same technology and during which capital is accumulated.

Now the Pareto distribution is stochastically decreasing in its parameter, i.e., a distribution with a low parameter is more valuable than one with a higher parameter. Also, a higher elasticity of output with respect to the input capital enhances the contribution of this factor. The model of this chapter asserts that a low parameter for the Pareto distribution coupled with a high value for the elasticity of output with respect to the capital input could generate sustained growth. We have also shown that as the economy embarks on a sustained growth path, more and more resources are devoted to the sequential search. In its advanced state, the ever increasing amounts of capital used up in the search process more than compensate for the diminishing returns in R&D in the sense that the lower bound on the probability of failure tends to zero as the economy's technological level tends to infinity.

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Appendix1

The sequential search

Suppose that when period t begins the technological level of the high-technology good sector is a_t and its capital stock is K_{yt}. Each firm i in the high-technology good sector thus begins the sequential stage of period t with $a_{it} = a_t$ as its technological level and $k_{it} = K_{yt} / | I |$ as the amount of capital it has managed to raise in the capital allocation stage of this period. Let $G(w)$ be the distribution function of the wage rate that a firm i expects to prevail in the production stage of period t.

Let us situate ourselves at a particular instant during the sequential stage of period t and consider a particular firm in the high-technology good sector. Suppose that the highest technological level at the disposal of this firm is a and its remaining capital is k. The question we need to answer is whether the firm should continue or terminate its sequential search. To answer this question, we now proceed with a dynamic programming argument.

A1.1. The case $0 < k < \varepsilon$

When $0 < k < \varepsilon$, the optimal decision for the firm is to terminate the sequential search and invest the remaining capital in the technology with productivity a. Given the wage rate w, the profit associated with this decision is

$$\alpha\left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} a^{1/\alpha} k .$$

The expected profit of the firm, given its expectations about the wage rate prevailing in the production and consumption stage of period t is thus given by

$$v_1(a,k\,|\,G) = \int \alpha\left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} a^{1/\alpha} k dG(w)$$

$$= \alpha a^{1/\alpha} k \int \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} dG(w)$$

$$= v_1(a,k) \int \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} dG(w), \qquad 0 < k < \varepsilon.$$

Where we have let

$$v_1(a,k) = \alpha a^{1/\alpha} k, \qquad 0 < k < \varepsilon.$$

A1.2. The case $\varepsilon < k < 2\varepsilon$

When $\varepsilon < k < 2\varepsilon$, the firm can take one more technology draw without reducing its capital stock to 0. If it terminates the sequential search, then its expected profit is given by

(A1.2.1) $$\alpha a^{1/\alpha} k \int \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} dG(w).$$

On the other hand, if it decides to take another technology draw, then its expected profit is given by

(A1.2.2)
$$F(a)v_1(a,k-\varepsilon\,|\,G) + \int_a^\infty v_1(a',k-\varepsilon\,|\,G)dF(a')$$
$$= [F(a)v_1(a,k-\varepsilon) + \int_a^\infty v_1(a',k-\varepsilon)dF(a')] \int \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} dG(w).$$

In (A1.2.2), the first term on the left side captures the component of the expected profit under the event that the technology draw fails to yield a better technology. This event occurs with probability $F(a)$ and conditioned on this event the expected profit of the firm is $v_1(a,k-\varepsilon\,|\,G)$. The second term on the left of (A1.2.2) captures the component of the expected profit under the event that the technology

draw results in a new technology with productivity $a' > a$. Given $a' > a$, the expected profit is $v_1(a', k-\varepsilon \mid G)$.
The optimal expected profit of the firm is then

$$\begin{aligned}
& v_2(a, k-\varepsilon \mid G) \\
&= \max\left\{[F(a)v_1(a,k-\varepsilon) + \int_a^\infty v_1(a',k-\varepsilon \mid G)dF(a')], \alpha a^{1/\alpha} k\right\} \int \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} dG(w) \\
&= \max\left\{[F(a)\alpha a^{1/\alpha}(k-\varepsilon) + \int_a^\infty \alpha (a')^{1/\alpha}(k-\varepsilon)dF(a')], \alpha a^{1/\alpha} k\right\} \int \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} dG(w) \\
&= \alpha k \max\left\{(1-\frac{\varepsilon}{k})[F(a)a^{1/\alpha} + \int_a^\infty (a')^{1/\alpha} dF(a')], a^{1/\alpha}\right\} \int \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} dG(w) \\
&= \alpha k \max\left\{\varphi_2(a,k), a^{1/\alpha}\right\} \int \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} dG(w) \\
&= v_2(a,k) \int \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} dG(w),
\end{aligned}$$

Where we have defined

$$\text{(A1.2.3)} \qquad \begin{aligned} \varphi_2(a,k) &= (1-\frac{\varepsilon}{k})[F(a)a^{1/\alpha} + \int_a^\infty (a')^{1/\alpha} dF(a')] \\ v_2(a,k) &= \alpha k \max\left\{\varphi_2(a,k), a^{1/\alpha}\right\} \end{aligned}$$

Now let $m_2(a,k) = \varphi_2(a,k) - a^{1/\alpha}$, Then

$$\text{(A1.2.4)} \qquad \begin{aligned} \frac{\partial m_2(a,k)}{\partial a} &= \frac{\partial(\varphi_2(a,k) - a^{1/\alpha})}{\partial a} \\ &= [(1-\frac{\varepsilon}{k})F(a) - 1]\frac{a^{(1-\alpha)/\alpha}}{\alpha} < \frac{-\varepsilon}{k}\frac{a^{(1-\alpha)/\alpha}}{\alpha} < \frac{-\varepsilon}{\alpha k} \leq \frac{-1}{2\alpha} \end{aligned}$$

Because the partial derivative $\frac{\partial m_2(a,k)}{\partial a}$ is negative and bounded above by $\frac{-1}{2\alpha}$, it is clear that (A1.2.2) will be negative—and thus the firm will not take another technology draw—if a is large. When a = 1, we have

(A1.2.5) $$m_2(1,k) = \frac{1-\alpha\varepsilon\lambda/k}{-1+\alpha\lambda}.$$

To insure the convergence of the integral in (A1.2.3), we must and will assume that $-1+\alpha\lambda > 0$. Furthermore, depending on the values of $\alpha, \lambda, \varepsilon,$ and k, the right side of the last equality in (A1.2.5) might be positive, negative, or equal to 0. Note that if $-1+\alpha\lambda > 0$ then $m_2(1,k) < 0$ when k is in a right neighborhood of ε. If $m_2(1,k) \geq 0$, there exists a unique value of $a \geq 1$ such that $m_2(1,k) = 0$. Such a value of a satisfies the following equation:

(A1.2.6) $$\varphi_2(a,k) - a^{1/\alpha} = 0,$$

and will be denoted by $\overline{a}(k)$. When $m_2(1,k) < 0$ there is no value of a that satisfies (A1.2.1). In this case, we set $\overline{a}(k)$=1. We can extend $\overline{a}(k)$ in the obvious way to $0 < k < 2\varepsilon$ by setting $\overline{a}(k)$=1 for $0 < k < \varepsilon$. As defined, $\overline{a}(k)$, $0 < k < 2\varepsilon$ represent threshold technological level such that the sequential search will continue as long as the highest technological level under the firm's command falls short of $\overline{a}(k)$, i.e., as long as $a<\overline{a}(k)$, where k is the firm's remaining capital. Observe that the curve $a \to m_2(a,k)$, which has been shown to be strictly downward sloping, shifts upward when k increases. Hence the curve $k \to \overline{a}(k), 0 < k < 2\varepsilon$, is increasing and will be strictly increasing once it has risen over 1. As already explained, $m_2(1,k) < 0$ when k is in a right neighborhood of ε. Hence $\overline{a}(k)$=1 for k in a right neighborhood of ε. The following figure depicts the behavior of $k \to \overline{a}(k), 0 < k < 2\varepsilon$.

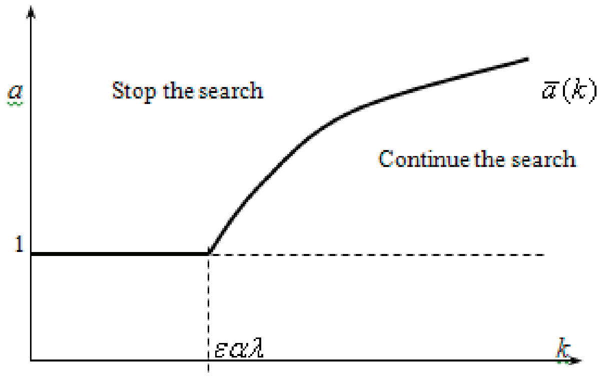

Figure 3. The threshold technological level $\overline{a}(k)$

A1.3. The case $2\varepsilon < k < 3\varepsilon$

When $2\varepsilon < k < 3\varepsilon$, the firm can take one more technology draw without reducing its capital stock to 0. If it terminates the sequential search, then its expected profit is given by

$$\alpha a^{1/\alpha} k \int \left(\frac{1-\alpha}{w} \right)^{(1-\alpha)/\alpha} dG(w) .$$

On the other hand, if it decides to take another technology draw, then its expected profit is given by

$$F(a) v_2 (a, k-\varepsilon) \mid G) + \int_a^{\infty} v_2 (a', k-\varepsilon \mid G) dF(a')$$

$$= [F(a) v_2 (a, k-\varepsilon) + \int_a^{\infty} v_2 (a', k-\varepsilon) dF(a')] \int \left(\frac{1-\alpha}{w} \right)^{(1-\alpha)/\alpha} dG(w)$$

Hence the firm's optimal expected profit is given by

$$v_3(a,k\,|\,G) = \max\left\{[F(a)v_2(a,k-\varepsilon) + \int_a^{\infty} v_2(a',k-\varepsilon)dF(a')], \alpha a^{1/\alpha}k\right\}$$

$$\int\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} dG(w),$$

$$= \max\left\{\begin{array}{l}[F(a)\alpha(k-\varepsilon)\max\{\varphi_2(a,k-\varepsilon),a^{1/\alpha}\}+ \\ \int_a^{\infty}\alpha(k-\varepsilon)\max\{\varphi_2(a',k-\varepsilon),(a')^{1/\alpha}\}dF(a')], \alpha a^{1/\alpha}k\end{array}\right\}$$

$$\int\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} dG(w),$$

$$= \alpha k \max\left\{\begin{array}{l}(1-\frac{\varepsilon}{k})[F(a)\max\{\varphi_2(a,k-\varepsilon),a^{1/\alpha}\}+ \\ \int_a^{\infty}\max\{\varphi_2(a',k-\varepsilon),(a')^{1/\alpha}\}dF(a')], a^{1/\alpha}\end{array}\right\}$$

$$\int\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} dG(w),$$

$$= \alpha k \max\{\varphi_3(a,k-\varepsilon),a^{1/\alpha}\}\int\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} dG(w),$$

$$= v_3(a,k)\int\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} dG(w),$$

where we have defined

$$\varphi_3(a,k-\varepsilon) = (1-\frac{\varepsilon}{k})[F(a)\max\{\varphi_2(a,k-\varepsilon),a^{1/\alpha}\}$$

$$+\int_a^{\infty}\max\{\varphi_2(a',k-\varepsilon),(a')^{1/\alpha}\}dF(a')]$$

$$v_3(a,k) = \max\{\varphi_3(a,k-\varepsilon),a^{1/\alpha}\}.$$

Observe that $\varphi_3(a,k-\varepsilon)$, $a \geq 1$, $2\varepsilon < k < 3\varepsilon$ is continuous in (*a, k*). It is also strictly increasing in both *a* and *k*. Its partial derivative with respect to *a* exists at any point (*a, k*) with $a > 1$ except at the point $(\overline{a}(k),k)$ where $\overline{a}(k) > 1$.

Now define

$$m_3(a,k) = \varphi_3(a,k-\varepsilon) - a^{1/\alpha}$$

Next, observe that

$$\max\{\varphi_2(a,k-\varepsilon), a^{1/\alpha}\} = \varphi_2(a,k-\varepsilon) \text{ if } a < \overline{a}(k-\varepsilon),$$
$$= a^{1/\alpha}, \text{otherwise.}$$

Hence for $a < \overline{a}(k-\varepsilon)$, we have

(A1.3.1)
$$\begin{aligned}\frac{\partial m_3(a,k)}{\partial a} &= (1-\frac{\varepsilon}{k})(F'(a)\varphi_2(a,k-\varepsilon) + F(a)\frac{\partial \varphi_2(a,k-\varepsilon)}{\partial a} - \varphi_2(a,k-\varepsilon) - \frac{a^{(1-\alpha)/\alpha}}{\alpha} \\ &= [(1-\frac{\varepsilon}{k})F(a)\frac{\partial \varphi_2(a,k-\varepsilon)}{\partial a} - \frac{a^{(1-\alpha)/\alpha}}{\alpha} \\ &< \frac{\partial \varphi_2(a,k-\varepsilon)}{\partial a} - \frac{a^{(1-\alpha)/\alpha}}{\alpha} \\ &< \frac{-1}{2\alpha}\end{aligned}$$

where the last inequality in (A1.3.1) has been obtained with the help of (A1.2.4).
For $a > \overline{a}(k-\varepsilon)$, we have

(A1.3.2)
$$\begin{aligned}\frac{\partial m_3(a,k)}{\partial a} &= \alpha k\left((1-\frac{\varepsilon}{k})F(a)\frac{a^{(1-\alpha)/\alpha}}{\alpha} - \frac{a^{(1-\alpha)/\alpha}}{\alpha}\right) \\ &= ka^{(1-\alpha)/\alpha}((1-\frac{\varepsilon}{k})F(a)-1) < -\varepsilon a^{(1-\alpha)/\alpha} < -\varepsilon.\end{aligned}$$

According to (A1.3.1) and (A1.3.2) $\partial m_3(a,k)/\partial a$ is negative and uniformly bounded above for all $a \neq \overline{a}(k)$. Hence $a \to m_3(a,k)$ is strictly decreasing and tends to -∞ when a tends to +∞. Thus if $m_3(a,k) = 0$ for some a, then this value of a, say $\overline{a}(k)$ must be unique and satisfies the following relation:

(A1.3.3)
$$\varphi_3(a,k-\varepsilon) - a^{1/\alpha} = 0.$$

If no value of a satisfies (A1.3.3), then set $\overline{a}(k)=1$. Observe that as a function of *a*, the left side of (A1.3.3) shifts upward as *k* increases. Hence $\overline{a}(k), 2\varepsilon < k \le 3\varepsilon$ is increasing with *k*. Now note that

$$\text{(A1.3.4)} \quad \lim_{k\to 2\varepsilon}[\varphi_3(a,k-\varepsilon)-a^{1/\alpha}]=\frac{1}{2}\left(\begin{matrix} F(a)\max\{\varphi_2(a,k-\varepsilon),a^{1/\alpha}\} \\ +\int_a^\infty \max\{\varphi_2(a',k-\varepsilon),(a')^{1/\alpha}\}dF(a')] \end{matrix}\right)-a^{1/\alpha}$$

Furthermore, when $k=\varepsilon$, we have

$$\text{(A1.3.5)} \quad \varphi_2(a,k-\varepsilon)=(1-\frac{\varepsilon}{k})\left(F(a)\,a^{1/\alpha}+\int_a^\infty (a')^{1/\alpha}dF(a')\right)=0\,.$$

Using (A1.3.5), we can rewrite (A1.3.4) as

$$\lim_{k\to 2\varepsilon}[\varphi_3(a,k-\varepsilon)-a^{1/\alpha}]=\frac{1}{2}\left(F(a)\,a^{1/\alpha}+\int_a^\infty (a')^{1/\alpha}dF(a')\right)-a^{1/\alpha}\,,$$

which is nothing other than the form assumed by (A1.2.6) when $k=2\varepsilon$. Therefore,

$$\lim_{k\to 2\varepsilon}\overline{a}(k)=\overline{a}(2\varepsilon)\,.$$

Therefore, the curve $a\to\overline{a}(k), 0<k<3\varepsilon$, is increasing and continuous.

A1.4. The General Case $n\varepsilon < k < (n+1)\varepsilon, n>0$

To find the solution to the general sequential search problem of the firm, let us now situate ourselves at a particular instant during the sequential search stage and suppose that (*a, k*) is the state of the search at this instant. Conditioned on this state and its expectations about the wage rate that will prevail when the search is over, let $v(a,k\,|\,G)$ be the optimal expected profit for the firm. Given the remaining capital *k*, the maximum number of technology draws that the firm can take is Ceiling[k/ε], where Ceiling[*x*] is the smallest integer greater than or equal to *x*. We have already solve the optimal search when Ceiling[k/ε] is less than or equal to 3. Now let $n > 3$ be a given integer, and suppose that the optimal solution to the sequential search has been found for all the states (*a, k*) with Ceiling[k/ε] < *n*. Then proceeding as in previous case, we can assert that the optimal expected profit for the firm under a state (*a, k*) with Ceiling[k/ε]= *n* is given by

$$v(a,k\mid G) = v_n(a,k\mid G) = \max\left\{[F(a)v_{n-1}(a,k-\varepsilon) + \int_a^\infty v_{n-1}(a',k-\varepsilon)dF(a')], \alpha a^{1/\alpha}k\right\}$$
$$\int\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} dG(w),$$
$$= \max\left\{\begin{matrix}[F(a)\alpha(k-\varepsilon)\max\{\varphi_{n-1}(a,k-\varepsilon),a^{1/\alpha}\} + \\ \int_a^\infty \alpha(k-\varepsilon)\max\{\varphi_{n-1}(a',k-\varepsilon),(a')^{1/\alpha}\}dF(a')],\alpha a^{1/\alpha}k\end{matrix}\right\}$$
$$\int\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} dG(w),$$

$$= \alpha k \max\left\{\begin{matrix}(1-\frac{\varepsilon}{k})[F(a)\max\{\varphi_{n-1}(a,k-\varepsilon),a^{1/\alpha}\} + \\ \int_a^\infty \max\{\varphi_{n-1}(a',k-\varepsilon),(a')^{1/\alpha}\}dF(a')], a^{1/\alpha}\end{matrix}\right\}$$
$$\int\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} dG(w),$$
$$= \alpha k \max\{\varphi_n(a,k-\varepsilon), a^{1/\alpha}\}\int\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} dG(w),$$
$$= v_n(a,k)\int\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} dG(w),$$

where we have defined

$$\varphi_n(a,k) = (1-\frac{\varepsilon}{k})[F(a)\max\{\varphi_{n-1}(a,k-\varepsilon),a^{1/\alpha}\} + \int_a^\infty \max\{\varphi_{n-1}(a',k-\varepsilon),(a')^{1/\alpha}\}dF(a')], \quad \text{(A1.4.1)}$$

$$v_n(a,k) = \alpha k \max\{\varphi_n(a,k), a^{1/\alpha}\}.$$

Now define

$$m_n(a,k) = \varphi_n(a,k) - a^{1/\alpha}.$$

be shown that $a \to m_n(a,k)$ is strictly decreasing and tends to $-\infty$ when a tends to $+\infty$.

Thus if $m_n(a,k) = 0$ for some a, then this value of a, say $\bar{a}(k)$ must be unique and satisfies the following relation:

(A1.4.2) $$\varphi_n(a,k) - a^{1/\alpha} = 0$$

If no value of a satisfies (A1.4.2), then set $\bar{a}(k)=1$. As defined, $\bar{a}(k), n\varepsilon < k < (n+1)\varepsilon, n > 0$ represent the threshold technological level such that the sequential search will continue as long as the highest technological level under the firm's command falls short of $\bar{a}(k)$, i.e., as long as $a < \bar{a}(k)$ where k is the firm's remaining capital. Again, it can then be shown that the curve $a \to m_n(a,k), 0 < k < n\varepsilon$ is continuous and increasing.

Observe that the expression inside the square brackets on the right side of (A1.4.1) is strictly greater than $a^{1/\alpha}$ Thus for any given a, the left side of (A1.4.2) will be strictly positive when k is sufficiently large, i.e., $\bar{a}(k)>1$ when k is large. This last result together with the fact that $\bar{a}(k)$ is strictly increasing in k once it rises above 1 then imply that $\bar{a}(k)$ rises monotonically to $+\infty$ as k $\to+\infty$.Using the functional forms of the production and search technologies, we can find the explicit form of the threshold technological level $\bar{a}(k)$ as follows.

Let k be a capital stock level such that $\bar{a}(k) > 1$.Then the firm will be indifferent between continuing and stopping the search when ($\bar{a}(k)$, k) is the state of its sequential search and the following equality must hold

$$\alpha(\bar{a}(k))^{1/\alpha} k \int \left(\frac{1-\alpha}{w} \right)^{(1-\alpha)/\alpha} dG(w) = F(\bar{a}(k)) v(\bar{a}(k), k-\varepsilon \mid G) + \int_{\bar{a}(k)}^{\infty} v(a', k-\varepsilon \mid G) da' \quad \text{(A1.4.3)}$$

Observe that the left side of (A1.4.3) represents the firm's expected profit if it stops the search, while the right side represents its expected profit if it chooses to take another technology draw. Because the threshold technological level is increasing in remaining capital and because $\bar{a}(k) > 1$, we must have $\bar{a}(k-\varepsilon) < \bar{a}(k)$. Hence the firm will enter the region of no R&D after its remaining capital is reduced to $k-\varepsilon$. Using this fact to evaluate the right side of (A1.4.3) we can rewrite this expression as follows:

(A1.4.4) $$(\bar{a}(k))^{1/\alpha} k = F(\bar{a}(k)) \bar{a}(k)^{1/\alpha} (k-\varepsilon) + (k-\varepsilon) \int_{\bar{a}(k)}^{\infty} (a')^{1/\alpha} dF(a').$$

An alternative to (A1.4.4) as a representation of the technological threshold is given by the following curve:

$$k : a \to \bar{k}(a) = \varepsilon(1 + (\alpha\lambda - 1)a^{\lambda}), a > 1.$$